Eigenvalues and Minimal surfaces

Mikhail Karpukhin

(UCL)

Minimal surfaces

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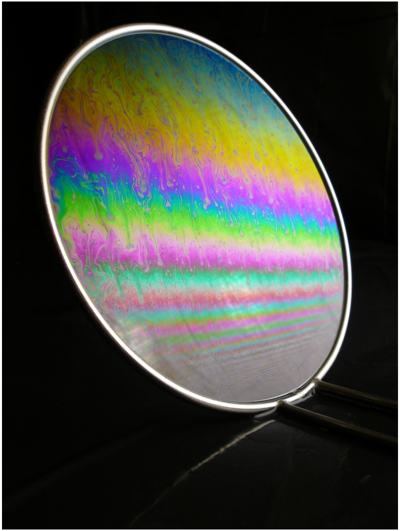

Figure: Credit: https://www.soapbubble.dk/en

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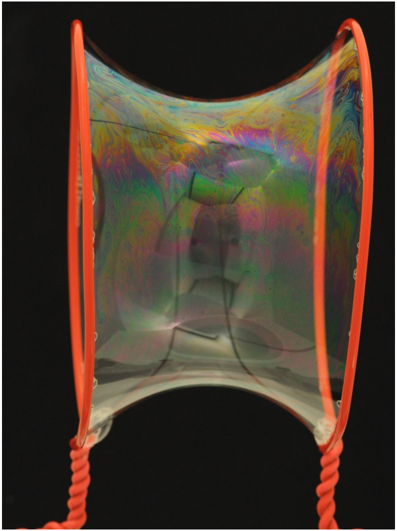

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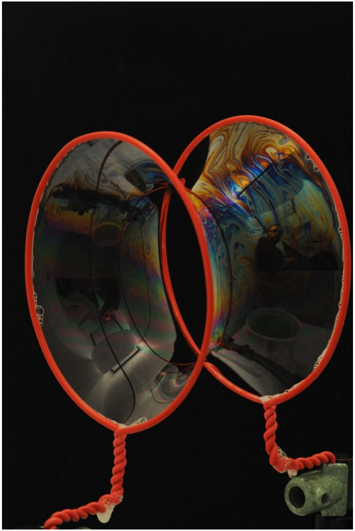

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Minimal surfaces locally minimize area. Equivalently $H = 0$.

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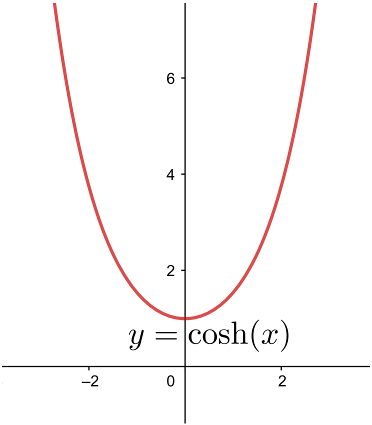

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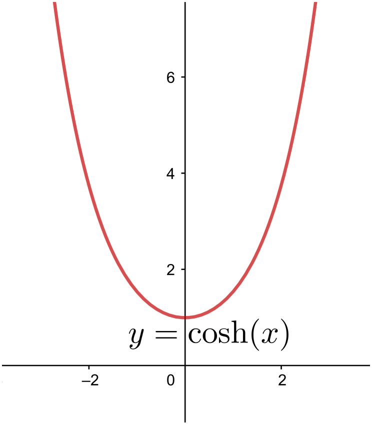

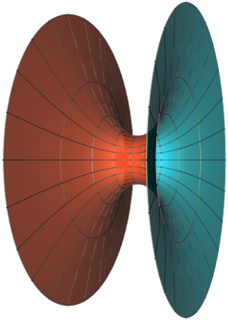

Figure: Credit: M. Weber's Minimal surfaces gallery

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  On $\mathbb{S}^1 \times \mathbb{S}^1$ it is parametrized as

$$\frac{1}{\sqrt{2}}\left(e^{i\phi}, e^{i\theta}\right)$$

Free boundary minimal surfaces in $\mathbb{B}^{n+1}$

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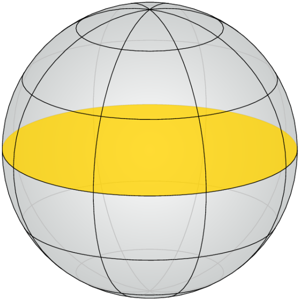

Figure: Equatorial disk

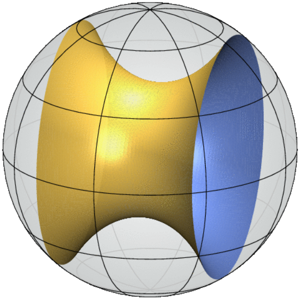

Figure: Critical catenoid

Pictures by Mario Schulz: https://mbschulz.github.io

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Set

$$\bar{\lambda}_k(M, g) = \lambda_k(M, g)\,\mathrm{Area}(M, g).$$

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Korevaar (1993): $\Lambda_k(M) < \infty$.

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Nadirashvili 1996; El Soufi, Ilias 2008:

Critical metrics $\longleftrightarrow$ Minimal surfaces in $\mathbb{S}^n$

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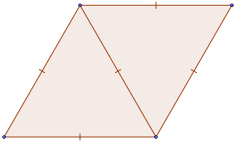
- Nadirashvili (1996): $\Lambda_1(\mathbb{T}^2) = \dfrac{8\pi^2}{\sqrt{3}}$ and the maximum is achieved on the *flat equilateral torus*.



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- $\mathbb{RP}^2$: *Veronese immersion* $v : \mathbb{RP}^2 \to \mathbb{S}^4$

$$v(x, y, z) = \left(xy, xz, yz, \frac{\sqrt{3}}{2}(x^2 - y^2), \frac{1}{2}(x^2 + y^2) - z^2 \right)$$

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- K.-Stern, 2020: any "reasonable" maximal metric has to be smooth.

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The goal is to find

$$\Sigma_1(N) = \sup_g \bar{\sigma}_1(N, g) = \sup_g \sigma_1(N, g) \mathrm{Length}_g(\partial N);$$

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Eigenvalues are

$$0, 1, 1, 2, 2, 3, 3, \ldots$$

Theorem of Weinstock

Theorem (Weinstock 1954) If N is simply connected, then

$$\Sigma_1(N) = 2\pi.$$

The equality is achieved for the unit disk $\mathbb{D}$.

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Fraser-Schoen (2014):

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- $\Sigma_1(\mathbb{A})$ is achieved on a critical catenoid;
- $\Sigma_1(\mathbb{M})$ is achieved on a critical Möbius band in $\mathbb{B}^4$.

Examples

All the following pictures are by Mario Schulz:
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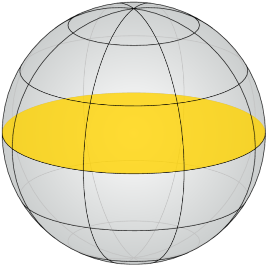

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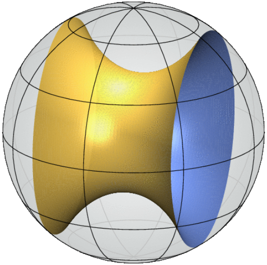

Figure: Critical catenoid

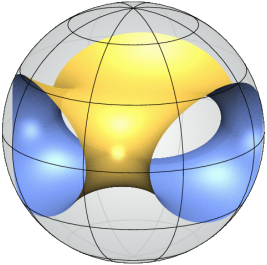

Figure: TBN

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- Kao-Osting-Oudet (2020): numerics for low number of boundary components:

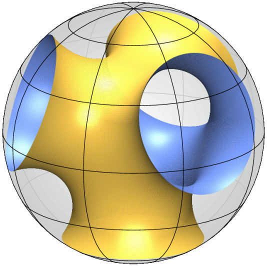

Figure: Tetrahedron

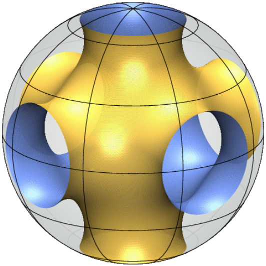

Figure: Octahedron

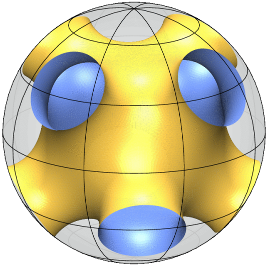

Figure: Skew cube

More pictures

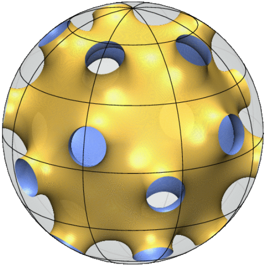

Figure: 20 bc

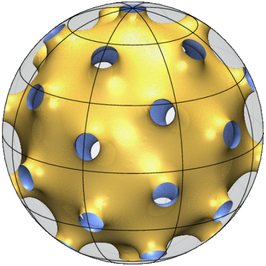

Figure: 32 bc

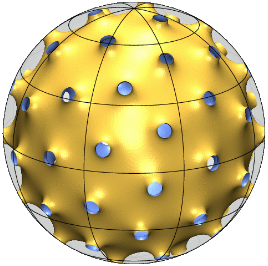

Figure: 61 bc

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  The boundary measures converge to twice the surface measure.

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- There exists $c = c(M) > 0$ such that

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Steklov $\longrightarrow$ Laplace

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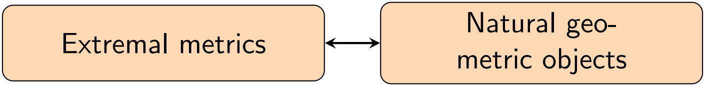
Meta-question

For which (eigenvalue) functionals one has



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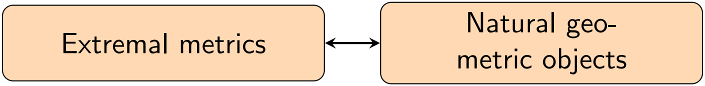

Very unexplored with many accessible problems.

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3. Many open questions for higher dimensional manifolds.

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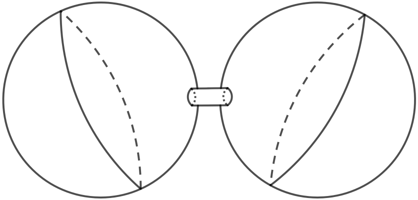

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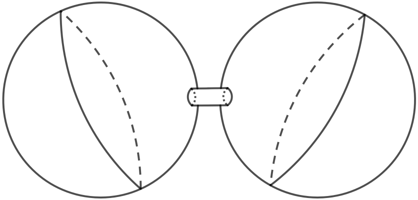

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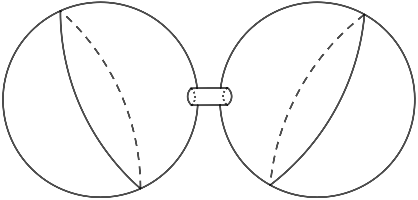

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- Nadirashvili–Penskoi (2018):
  $\Lambda_2(\mathbb{RP}^2) = \Lambda_1(\mathbb{RP}^2) + \Lambda_1(\mathbb{S}^2) = 20\pi$.

Higher eigenvalues: new results

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- K. (2019):

$$\Lambda_k(\mathbb{RP}^2) = 4\pi(2k+1)$$